Divergence Free Wavelets with Flexible Directional Localization

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Divergence free wavelets

- Directional selectivity:
Divergence free wavelets

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- Multi-scale representation:
Divergence free wavelets

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I. Approximation of functions
Approximation of functions
Approximation of functions
Approximation of functions

Let \( f \in C^k([0,1]) \cap L_2([0,1]) \). Then the linear N-term Fourier series approximation \( \tilde{f}_N \) satisfies:

\[
\| f - \tilde{f}_N \| \leq C \cdot 2^{-2k}
\]
Approximation of functions
Approximation of functions

\[ |\hat{f}(n)| \]
Approximation of functions
Approximation of functions
Approximation of functions

\[ f(n) \sim \frac{1}{n^k} \quad (k > 1) \]
Approximation of functions
Approximation of functions

\[ f(x) \]
Approximation of functions
Approximation of functions

| \( \hat{f}(n) \) |
Approximation of functions
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\[ \hat{\psi}_j \]
Approximation of functions

Calderon condition: \[ \sum_{j=-\infty}^{\infty} |\hat{\psi}_j(\xi)|^2 = 1, \ \forall \xi \]
Approximation of functions
Approximation of functions
Approximation of functions

\[ 2^j \pi \]
Approximation of functions
Approximation of functions
Approximation of functions
Approximation of functions

\[ \psi_{0,k}(x) \]
Approximation of functions

\[\psi_{0, k}(x)\]
Approximation of functions

\[ \psi_{1,k}(x) \]
Approximation of functions

$\psi_{2,k}(x)$
Approximation of functions

$\psi_{3,k}(x)$
Approximation of functions

\[ \xi = 2^3 \]
\[ \bar{x} = 2^{-j} k \]

\[ \psi_{3,k}(x) \]
Approximation of functions
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Sparse representation: $f(x) \approx \sum_{j,k \in \mathcal{I}_f} f_{jk} \psi_{jk}(x)$
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Approximation of functions

Mallat (2009):

**Theorem 9.12.** If $f$ has $K$ discontinuities on $[0, 1]$ and is uniformly Lipschitz $\alpha$ between these discontinuities, with $1/2 < \alpha < q$, then

$$\varepsilon_l(M, f) = O(K \| f \|^2_{C^\alpha} M^{-1}) \quad \text{and} \quad \varepsilon_n(M, f) = O(\| f \|^2_{C^\alpha} M^{-2\alpha}).$$
Approximation of functions on $\mathbb{R}^n$
Approximation of functions on $\mathbb{R}^n$

- Use that $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$
Approximation of functions on $\mathbb{R}^n$

$$\hat{\psi}_j(\xi_1, \xi_2) = \hat{\psi}_j(\xi_1) \hat{\psi}_j(\xi_2)$$
Approximation of functions on $\mathbb{R}^n$

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Laplace operator:

$$\hat{\Delta} = -|\xi|^2$$
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Discontinuity:
Approximation of functions on $\mathbb{R}^n$
Approximation of functions on $\mathbb{R}^n$

$$\hat{\psi}_j(\xi_1, \xi_2) = \hat{\psi}_j(\xi_1) \hat{\psi}_j(\xi_2)$$
Approximation of functions on $\mathbb{R}^n$

$$\hat{\psi}_j(\xi_1, \xi_2) = \hat{\phi}_j(\xi_1) \hat{\psi}_j(\xi_2)$$

$$\hat{\psi}_j(\xi_1, \xi_2) = \hat{h}_j(|\xi|)$$
Approximation of functions on $\mathbb{R}^n$

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Approximation of functions on $\mathbb{R}^n$

$\hat{\psi}_j(\xi_1, \xi_2) = \phi_j(\xi_1) \hat{\psi}_j(\xi_2)$

$\hat{\psi}_j(\xi_1, \xi_2) = \hat{h}_j(|\xi|)$
Approximation of functions on $\mathbb{R}^n$

\begin{align*}
\hat{\psi}_j(\xi_1, \xi_2) &= \hat{\psi}_j(\xi_1) \hat{\psi}_j(\xi_2) \\
\hat{h}_j(|\xi|) \hat{\gamma}_t(\theta_\xi)
\end{align*}

Approximation of functions on $\mathbb{R}^n$

\[
\hat{\psi}_j(\xi_1, \xi_2) = \psi_j(\xi_1) \hat{\psi}_j(\xi_2)
\]

\[
\hat{\psi}_j(\xi_1, \xi_2) = \hat{h}_j(|\xi|) \hat{\gamma}_t(\theta_\xi)
\]

Approximation of functions on $\mathbb{R}^n$
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Candès & Donoho (2004):

**Theorem 1.3** Under the assumptions of Theorem 1.2, the $n$-term approximation $f_n^C$ obtained by simple thresholding in a curvelet frame achieves

$$\| f - f_n^C \|_{L_2}^2 \leq C \cdot n^{-2} \cdot (\log n)^3.$$
Approximation of functions on $\mathbb{R}^n$

- Fefferman [1973]
Approximation of functions on $\mathbb{R}^n$

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- Candès [1999, 2005]

https://statweb.stanford.edu/~candes/
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- Candes [1999, 2005]
- Do & Vetterli [2003], Kutyniok & Labate [2006]
- Unser et al. [2010, 2012]

https://statweb.stanford.edu/~candes/
Polar wavelets

\[ \hat{\psi}_j(\xi_1, \xi_2) = \hat{h}_j(|\xi|) \hat{\gamma}_t(\theta_\xi) \]
Polar wavelets

\[ \hat{\psi}_j(\xi_1, \xi_2) = \hat{h}_j(|\xi|) \hat{\gamma}_t(\theta_\xi) \]

\[ \hat{\gamma}(\theta_\xi) = \sum_m \beta_{j,t} e^{im\theta_\xi} \]
Polar wavelets

\[ \psi_{j,t}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi) e^{i\langle \xi, x \rangle} d\xi \]
Polar wavelets

\[
\psi_{j,t}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi) e^{i\langle \xi,x \rangle} \, d\xi
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}(|\xi|) \hat{\gamma}(\theta_\xi) e^{i\langle \xi,x \rangle} \, d\xi
\]
Polar wavelets

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\psi_{j,t}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi) e^{i\langle \xi, x \rangle} \, d\xi
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\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}_j(|\xi|) \hat{\gamma}(\theta_\xi) \left( \sum_{n=-\infty}^{\infty} i^n e^{in(\theta_\xi - \theta_\xi)} J_n (|\xi| \ |x|) \right) \, d\xi
\]
Polar wavelets

\[ \psi_{j,t}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi) e^{i\langle \xi, x \rangle} \, d\xi \]

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Jacobi-Anger formula
Polar wavelets

\[
\psi_{j,t}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi) e^{i\langle \xi, x \rangle} \, d\xi
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}(|\xi|) \hat{\gamma}(\theta_\xi) e^{i\langle \xi, x \rangle} \, d\xi
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}_j(|\xi|) \hat{\gamma}(\theta_\xi) \left( \sum_{n=-\infty}^{\infty} i^n e^{i n (\theta_x - \theta_\xi)} J_n(|\xi| |x|) \right) \, d\xi
\]

\[
\hat{\gamma}(\theta_\xi) = \sum_m \beta_m^{j,t} e^{im\theta_\xi}
\]
Polar wavelets

\[ \psi_{j,t}(x) = \sum_{m} i^m \beta_{j,t}^m e^{im\theta_x} h_m(|x|) \]
Polar wavelets

\[ \psi_{j,t}(x) = \sum_{m} i^m \beta_{m}^{j,t} e^{im\theta_{x}} h_{m}(|x|) \]
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Polar wavelets

$$\psi_{j,t}(x) = \sum_m i^m \beta_{j,t}^m e^{im\theta_x} h_m(|x|)$$
II. Divergence free wavelets
Divergence freedom

\[ \nabla \cdot \vec{u} = 0 \]
Divergence freedom

\[ \nabla \cdot \vec{u} = 0 \]

\[ \mathcal{F} \]
Divergence freedom

$$\nabla \cdot \vec{u} = 0$$

$$\mathcal{F}$$

$$\vec{\xi} \cdot \hat{\vec{u}} = 0$$
Divergence freedom

\[ \nabla \cdot \vec{u} = 0 \]

\[ \vec{\xi} \cdot \hat{\vec{u}} = 0 \]
Divergence freedom

\[ \nabla \cdot \vec{u} = 0 \quad \text{and} \quad \xi \cdot \hat{\xi} \cdot \vec{u} = 0 \]
Divergence freedom

\[ \nabla \cdot \tilde{\mathbf{u}} = 0 \]

\[ \xi \cdot \mathbf{\hat{u}} = 0 \]
Divergence freedom

\[ \nabla \cdot \vec{u} = 0 \]

\[ \hat{\xi} \cdot \hat{\vec{u}} = 0 \]
Divergence free polar wavelets

- Divergence free basis:
Divergence free polar wavelets

- Divergence free basis:
Divergence free polar wavelets

- Divergence free basis:

\[ \hat{\psi}_s(\xi) = \hat{\psi}_s(\xi) \vec{e}_\theta \]
Divergence free polar wavelets

\[ \hat{\psi}_s(\xi) = \psi(|\xi|) \bar{e}_\theta \]
Divergence free polar wavelets

- Divergence free basis:

\[ \hat{\psi}_s(\xi) = \hat{\psi}(|\xi|) \bar{e}_\theta \]
Divergence free polar wavelets

- Divergence free basis:

\[ \hat{\psi}_s(\xi) = \gamma(\theta_\xi) \hat{\psi}(|\xi|) \vec{e}_\theta \]
Divergence free polar wavelets

Proposition 1. Let $U_j$ be the $(M_j \times 2N_j + 1)$-dimensional matrix formed by the angular localization coefficients $\beta_n^{j, t} = \beta_n^j e^{-i n t (2\pi / M_j)}$ for the $M_j$ different orientations, and let $D_j$ be a diagonal matrix of size $(2N_j + 1) \times (2N_j + 1)$. When the Caldèron admissibility condition $\sum_{j \in \mathbb{Z}} |\hat{h}(2^{-j} |\xi|)|^2 = 1, \forall \xi \in \mathbb{R}^2$ is satisfied and $U_j^H U_j = D_j$ with $\text{tr}(D_j) = 1$ for all levels $j$, then any divergence free vector field $\vec{u}(x) \in L_2^{\text{div}}(\mathbb{R}^{2,2})$ has the representation

$$\vec{u}(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \sum_{t=1}^{M_j} \langle \vec{u}(y), \vec{\psi}_{j, k, t}(y) \rangle \vec{\psi}_{j, k, t}(x)$$  \hspace{1cm} (5a)

with frame functions

$$\vec{\psi}_{j, k, t}(x) = \frac{2^j}{2\pi} \vec{\psi}(R_{2\pi t / M_j}(2^j x - k))$$  \hspace{1cm} (5b)

where $\vec{\psi}(x)$ is given by Eq. 4 and $R_{2\pi t / M_j}$ is the rotation by $2\pi t / M_j$. 
Scalar free polar wavelets

\[ \psi_{j,t}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi) e^{i\langle \xi, x \rangle} \, d\xi \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}(|\xi|) \hat{\gamma}(\theta_\xi) e^{i\langle \xi, x \rangle} \, d\xi \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}_j(|\xi|) \hat{\gamma}(\theta_\xi) \left( \sum_{n=-\infty}^\infty i^n e^{in(\theta_x - \theta_\xi)} J_n(|\xi| \, |x|) \right) \, d\xi \]
Scalar free polar wavelets

\[ \psi_{j,t}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi) e^{i\langle \xi, x \rangle} \, d\xi \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}(|\xi|) \hat{\gamma}(\theta_\xi) e^{i\langle \xi, x \rangle} \, d\xi \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}_j(|\xi|) \hat{\gamma}(\theta_\xi) \left( \sum_{n=-\infty}^{\infty} i^n e^{in(\theta_x - \theta_\xi)} J_n(|\xi| |x|) \right) \, d\xi \]

\[ \vec{e}_\theta = (-\sin \theta_\xi, \cos \theta_\xi)^T \]
Divergence free polar wavelets

\[ \hat{\gamma}(\xi) \vec{e}_\theta = \sum_{m} \beta_m e^{im\theta_\xi} \begin{pmatrix} -\sin \theta_\xi \\ \cos \theta_\xi \end{pmatrix} \]
Divergence free polar wavelets

\[ \hat{\gamma}(\xi) \hat{e}_\theta = \sum_m \beta_m e^{im\theta_\xi} \begin{pmatrix} -\sin \theta_\xi \\ \cos \theta_\xi \end{pmatrix} \]

\[ = \sum_m \beta_m e^{im\theta_\xi} \begin{pmatrix} -\frac{i}{2}(e^{-i\theta} - e^{i\theta}) \\ \frac{1}{2}(e^{-i\theta} + e^{i\theta}) \end{pmatrix} \]
Divergence free polar wavelets

\[ \hat{\gamma}(\xi) \hat{e}_\theta = \sum_m \beta_m e^{im\theta_\xi} \begin{pmatrix} -\sin \theta_\xi \\ \cos \theta_\xi \end{pmatrix} \]

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\[ = \begin{pmatrix} -\frac{i}{2} \left( \sum_m \beta_m e^{i(m-1)\theta_\xi} - \sum_m \beta_m e^{i(m+1)\theta_\xi} \right) \\ \frac{1}{2} \left( \sum_m \beta_m e^{i(m-1)\theta_\xi} + \sum_m \beta_m e^{i(m+1)\theta_\xi} \right) \end{pmatrix} \]
Divergence free polar wavelets

\[ \tilde{\psi}(\xi) = \sum_m \beta_m e^{im\theta_\xi} \hat{h}(|\xi|) \vec{e}_{\theta_\xi} \]

\[ \mathcal{F}^{-1} \]

\[ \tilde{\psi}(x) = \frac{1}{2} \sum_{\sigma \in \{-1,1\}} \sum_m i^{m+\sigma} \beta_m e^{i(m+\sigma)\theta_x} h_{m+\sigma}(|x|) \left( \begin{array}{c} -\sigma \\ i \end{array} \right) \]
Divergence free polar wavelets
Divergence free polar wavelets

\[ F^{-1} \]
Divergence free polar wavelets

\[ \mathcal{F}^{-1} \]
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error with only isotropic wavelets

\[ j \leq 3 \]
Divergence free polar wavelets

error with anisotropic wavelets

\[ j \leq 3 \]
Divergence free polar wavelets

Proposition 2. Let \( \tilde{\mathbf{u}}(x) \in L^2_\text{div}(\mathbb{R}^2) \) be a \( C^2 \)-smooth divergence free vector field away from \( C^2 \) discontinuities with \( \mathcal{F}^{-1}(\tilde{\mathbf{u}}) \in \mathcal{E}^2(A) \) [5, Def. 1], which we write as \( \tilde{\mathbf{u}} \in \mathcal{E}^2_{\text{div}}(A) \). When the windows \( \hat{\gamma}_j(\theta_\xi) \) and \( \hat{h}(|\xi|) \) satisfy the admissibility conditions of second generation curvelets [5, Sec. 2], then the \( n \)-largest coefficient \( |u_s|_n \) in the coefficient sequence \( (|u_s|)_n \) satisfies

\[
\sup_{\tilde{\mathbf{u}} \in \mathcal{E}^2_{\text{div}}(A)} |u_s|_n \leq C \cdot n^{-3/2} (\log n)^{3/2}.
\] (8)
Divergence free polar wavelets
Divergence free polar wavelets
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Divergence free polar wavelets
Divergence free polar wavelets in $\mathbb{R}^3$
Divergence free polar wavelets in $\mathbb{R}^3$

- Divergence freedom:

$$\text{div}(\vec{u}) = 0 \iff \hat{u} \in TS^{n-1}$$
“Polar” wavelets in $\mathbb{R}^3$

$$\hat{\psi}(\xi) = \hat{h}(|\xi|) \hat{\gamma}(\bar{\xi})$$
“Polar” wavelets in $\mathbb{R}^3$

$$
\hat{\psi}(\xi) = \hat{h}(|\xi|) \hat{\gamma}(\bar{\xi})
= \hat{h}(|\xi|) \left( \sum_{l,m} \beta_{lm} y_{lm}(\bar{\xi}) \right)
$$
“Polar” wavelets in $\mathbb{R}^3$

$$\hat{\psi}(\xi) = \hat{h}(|\xi|) \hat{\gamma}(\tilde{\xi})$$

$$= \hat{h}(|\xi|) \left( \sum_{l,m} \beta_{lm} y_{lm}(\tilde{\xi}) \right)$$
“Polar” wavelets in $\mathbb{R}^3$

\[ \hat{\psi}(\xi) = \hat{h}(|\xi|) \hat{\gamma}(\bar{\xi}) \]

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“Polar” wavelets in $\mathbb{R}^3$

$$\psi_{j,t}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\psi}_j(\xi) e^{i\langle \xi, x \rangle} \, d\xi$$
“Polar” wavelets in $\mathbb{R}^3$

$$
\psi_{j,t}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3_\xi} \hat{\psi}_j(\xi) \, e^{i\langle \xi, x \rangle} \, d\xi
$$

$$
= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3_\xi} \hat{h}(|\xi|) \, \hat{\gamma}(\xi) \, e^{i\langle \xi, x \rangle} \, d\xi
$$
“Polar” wavelets in $\mathbb{R}^3$

$$\psi_{j,t}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\psi}_j(\xi) e^{i\langle \xi, x \rangle} \, d\xi$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{h}(|\xi|) \hat{\gamma}(\bar{\xi}) e^{i\langle \xi, x \rangle} \, d\xi$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} \hat{h}_j(|\xi|) \hat{\gamma}(\bar{\xi}) \left(4\pi \sum_{l,m} i^l y_{lm}(\bar{\xi}) y_{lm}(\bar{x}) j_l(|\xi| |x|)\right) \, d\xi$$
"Polar" wavelets in $\mathbb{R}^3$

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\psi_{j,t}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\psi}_j(\xi) e^{i\langle \xi, x \rangle} \, d\xi
$$

$$
= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{h}(|\xi|) \hat{\gamma}(\bar{\xi}) e^{i\langle \xi, x \rangle} \, d\xi
$$

$$
= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} \hat{h}_j(|\xi|) \hat{\gamma}(\bar{\xi}) \left(4\pi \sum_{l,m} i^l y_{lm}(\bar{\xi}) y_{lm}(\bar{x}) j_l(|\xi| |x|) \right) \, d\xi
$$

Rayleigh formula
"Polar" wavelets in $\mathbb{R}^3$

$$
\psi_{j,t}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\psi}_j(\xi) e^{i\langle \xi, x \rangle} d\xi
$$

$$
= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{h}(|\xi|) \hat{\gamma}(\tilde{\xi}) e^{i\langle \xi, x \rangle} d\xi
$$

$$
= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} \hat{h}_j(|\xi|) \hat{\gamma}(\tilde{\xi}) \left( 4\pi \sum_{l,m} i^l y_{lm}(\tilde{\xi}) y_{lm}(\tilde{x}) j_l(|\xi| |x|) \right) d\xi
$$

$$
\hat{\gamma}(\tilde{\xi}) = \sum_{lm} \beta_{lm}^{jt} y_{lm}(\tilde{\xi}) \quad \text{Rayleigh formula}
$$
“Polar” wavelets in $\mathbb{R}^3$

$$\hat{\psi}(\xi) = \hat{h}(|\xi|) \hat{\gamma}(\bar{\xi})$$

$$= \hat{h}(|\xi|) \left( \sum_{l,m} \beta_{lm} y_{lm}(\bar{\xi}) \right)$$

$$\quad \left| \mathcal{F}^{-1} \right.$$ $$\psi_{j,t}(x) = \sum_{l,m} i^l \beta_{lm} y_{lm}(\bar{x}) h_l(|\xi|)$$
“Polar” wavelets in $\mathbb{R}^3$
“Polar” wavelets in $\mathbb{R}^3$
“Polar” wavelets in $\mathbb{R}^3$
Divergence free polar wavelets in $\mathbb{R}^3$
Divergence free polar wavelets in $\mathbb{R}^3$
Divergence free polar wavelets in $\mathbb{R}^3$

$$\hat{\psi}(\xi) = \hat{h}(|\xi|) \hat{\gamma}(\bar{\xi}) \hat{e}_{\{\theta, \phi\}}(\bar{\xi})$$
Divergence free polar wavelets in $\mathbb{R}^3$

\[ \hat{\psi}(\xi) = \hat{h}(|\xi|) \hat{\gamma}(\tilde{\xi}) \hat{e}_{\{\theta, \phi\}}(\tilde{\xi}) \]
Divergence free polar wavelets in $\mathbb{R}^3$

$$\hat{\psi}(\xi) = \hat{h}(|\xi|) \hat{\gamma}(\bar{\xi}) \bar{e}_{\{\theta, \phi\}}(\bar{\xi})$$

$\bar{e}_{\theta}(\bar{\xi})$ is singular at pole
Divergence free polar wavelets in $\mathbb{R}^3$
Divergence free polar wavelets in $\mathbb{R}^3$
Divergence free polar wavelets in $\mathbb{R}^3$
Divergence free polar wavelets in $\mathbb{R}^3$
Divergence free polar wavelets in $\mathbb{R}^3$

- Hedgehog frame:

$$H = \left\{ \vec{e}_\phi(\xi_1), \vec{e}_\phi(\xi_2), \vec{e}_\phi(\xi_3) \right\}$$
Divergence free polar wavelets in $\mathbb{R}^3$

- Hedgehog frame:

$$H = \left\{ \vec{e}_\phi(\xi_1), \vec{e}_\phi(\xi_2), \vec{e}_\phi(\xi_3) \right\}$$

- Tight frame for $TS^2$
Divergence free polar wavelets in $\mathbb{R}^3$

- Hedgehog frame:

$$H = \left\{ \tilde{e}_\phi^1(\bar{\xi}), \tilde{e}_\phi^2(\bar{\xi}), \tilde{e}_\phi^3(\bar{\xi}) \right\}$$

- Tight frame for $TS^2$
Divergence free polar wavelets in $\mathbb{R}^3$

\[ \hat{\psi}^\kappa_{j,t}(\xi) = \hat{h}_j(|\xi|) \hat{\gamma}_{j,t}(\tilde{\xi}) \hat{e}^\kappa_{\phi}(\tilde{\xi}) \]
Divergence free polar wavelets in $\mathbb{R}^3$

**Proposition 4.** Let $w_{j,t}$ be the $(L_j + 1)^2$-dimensional vector formed by the rotated angular localization coefficients $\kappa_{l,m}^{i,t} = \sum_{m'} W(\lambda_t)^{m'}_{l,m} \kappa_{l,m'}$ for a localization window centered at $\lambda_t$, where $W(\lambda_t)^{m'}_{l,m}$ is the Wigner-D matrix implementing rotation in the spherical harmonics domain, and let $G^{l,m}$ be the $(L_j + 1)^2 \times (L_j + 1)^2$ dimensional matrix formed by the spherical harmonics product coefficients for fixed $(l,m)$. When the Caldèron condition $\sum_{j \in \mathbb{Z}} |h(2^{-j} |\xi|)|^2 = 1$, $\forall \xi \in \mathbb{R}^3$ is satisfied and $\delta_{l,0}\delta_{m,0} = \sum_{t=0}^{M_j} w_{j,t} G^{l,m} w_{j,t}$ (where $\delta_{i,j}$ is the Kronecker delta) then any $\vec{u}(x) \in L^2_{\text{div}}(\mathbb{R}^3)$ has the representation

$$
\vec{u}(x) = \sum_{a=1}^{3} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^3} \sum_{t=1}^{M_j} \langle \vec{u}(y), \vec{\psi}_{j,k,t}^{a}(y) \rangle \vec{\psi}_{j,k,t}^{a}(x)
$$

(12a)

with frame functions

$$
\vec{\psi}_{j,k,t}^{a}(x) = \frac{2^{3j/2}}{(2\pi)^{3/2}} \vec{\psi}_a(R_{\lambda_t}(2^j x - k)),
$$

(12b)

for $\vec{\psi}_a(x)$ defined in Eq. 11 and $R_{\lambda_t}$ the rotation from the North pole to $\lambda_t$. 

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Divergence free polar wavelets in $\mathbb{R}^3$
Divergence free polar wavelets in $\mathbb{R}^3$
Divergence free polar wavelets in $\mathbb{R}^3$
Divergence free polar wavelets in $\mathbb{R}^3$
III. The bigger picture
Polar wavelets in the plane
Polar wavelets in the plane
Polar wavelets in the plane

\[ \hat{\psi}_s \]

\[ h(|\xi|) \]

\[ \vec{e}_\theta \]

\[ h(|\xi|) \]
Polar wavelets in the plane
Polar wavelets in the plane
Polar wavelets in the plane
Polar wavelets in the plane
Polar wavelets in the plane
Polar wavelets in the plane

Helmholtz decomposition of vector field
Polar wavelets in the plane

Helmholtz decomposition of vector field
Polar wavelets in the plane

Helmholtz decomposition of vector field
Polar wavelets in the plane

Helmholtz decomposition of vector field

$\Omega_2^2(\mathbb{R}^2)$
Polar wavelets in the plane

\[ \Omega^1_\delta(\mathbb{R}^2) \xrightarrow{\text{d}} \Omega^2_\delta(\mathbb{R}^2) \]

Helmholtz decomposition of vector field
Polar wavelets in the plane

\[ \Omega_d^1(\mathbb{R}^2) \xrightarrow{\ast} \Omega_\delta^1(\mathbb{R}^2) \xrightarrow{\text{d}} \Omega_d^2(\mathbb{R}^2) \]

Hodge-Helmholtz decomposition
Polar wavelets in the plane

\[ \Omega^0_\delta(\mathbb{R}^2) \xrightarrow{d} \Omega^1_d(\mathbb{R}^2) \xrightarrow{\ast} \Omega^1_\delta(\mathbb{R}^2) \xrightarrow{d} \Omega^2_d(\mathbb{R}^2) \]

Hodge-Helmholtz decomposition
$\Psi_{\text{ec}}$: Local spectral exterior calculus

$$\Omega_0^0(\mathbb{R}^2) \xrightarrow{d} \Omega_1^1(\mathbb{R}^2) \xrightarrow{\ast} \Omega_0^0(\mathbb{R}^2) \xrightarrow{d} \Omega_1^2(\mathbb{R}^2)$$

---

$\Psi_{ec}$: Local spectral exterior calculus

$\Omega^0_\delta(S^2)$
\( \Psi_{ec}: \text{Local spectral exterior calculus} \)

\[ \Omega^0_\delta(S^2) \]

\[ \tilde{\gamma}_{j,t}(\bar{\xi}) \]
\[ \hat{\gamma}_{j,t}(\bar{\xi}) = \sum_{l,m} \beta_{lm} y_{lm}(\bar{\xi}) \]
$\Psi_{ec}: \text{Local spectral exterior calculus}$

$$\Omega_0^0(S^2) \quad \Omega_1^1(S^2)$$

$$\dot{\gamma}_{j,t}(\bar{\xi}) = \sum_{l,m} \beta_{lm} y_{lm}(\bar{\xi})$$
**Ψec: Local spectral exterior calculus**

\[ \Omega^0_\delta(S^2) \quad \Omega^1_\delta(S^2) \quad \Omega^1_d(S^2) \]

\[ \hat{\gamma}_{j,t}(\bar{\xi}) = \sum_{l,m} \beta_{lm} \psi_{lm}(\bar{\xi}) \]
\( \Psi_{\text{ec}}: \text{Local spectral exterior calculus} \)

\[
\Omega_{\delta}^0(S^2) \xrightarrow{\text{d}} \Omega_{\delta}^1(S^2) \xrightarrow{\star} \Omega_{\delta}^1(S^2) \xrightarrow{\text{d}} \Omega_{\delta}^2(S^2)
\]

\[
\hat{\gamma}_{j,t}(\bar{\xi}) = \sum_{l,m} \beta_{lm} y_{lm}(\bar{\xi})
\]
Summary
Summary

- Tight frames of divergence free wavelets
  - Flexible angular localization
  - Closed form expressions in spatial and frequency domain
  - Quasi optimal approximation properties in 2D
  - Intuitive correspondence to natural flow phenomena
Open Questions
Open Questions

- Compactly supported polar wavelets
Open Questions

- Compactly supported polar wavelets
Open Questions

- Compactly supported polar wavelets
Open Questions

- Compactly supported polar wavelets
- Boundary layer theory and curvelet-like anisotropic wavelets
- Approximation properties and implementation of 3D div-free wavelets