Mathematical Tools for Computer Graphics 2017

Assignment 1

Due date: 11/5/2017

1.) **Biorthogonal bases in $\mathbb{R}^2$:** We would like to construct the biorthogonal dual basis, if possible, for the vectors

$$\mathrm{u}_1 = a_1 \mathrm{e}_1 \tag{1a}$$

$$\mathrm{u}_2 = a_2 \mathrm{e}_2 \tag{1b}$$

for $a_1, a_2 \in \mathbb{R}$ with $\{\mathrm{e}_1, \mathrm{e}_2\}$ being the canonical basis for $\mathbb{R}^2$.

a.) When do u_1, u_2 span $\mathbb{R}^2$? Show this formally.

b.) Assuming u_1, u_2 span $\mathbb{R}^2$, construct the dual basis vectors $\tilde{\mathrm{u}}_1$, $\tilde{\mathrm{u}}_2$ such that

$$\langle \mathrm{u}_i, \tilde{\mathrm{u}}_j \rangle = \delta_{ij}.$$

c.) Write a python function that constructs the dual basis for given parameters a_1 a_2.

2.) **Frames in $\mathbb{R}^2$:** We would like to generalize the Mercedes Benz frame, given in $\mathbb{R}^2$ by

$$\mathrm{u}_1 = \mathrm{e}_2 \tag{2a}$$

$$\mathrm{u}_2 = -\frac{\sqrt{3}}{2}\mathrm{e}_1 - \frac{1}{2}\mathrm{e}_2 \tag{2b}$$

$$\mathrm{u}_3 = \frac{\sqrt{3}}{2}\mathrm{e}_1 - \frac{1}{2}\mathrm{e}_2, \tag{2c}$$

up to a possible rescaling of the vectors, to $\mathbb{R}^3$.

a.) Construct an analogue of the Mercedes Benz frame in $\mathbb{R}^3$. Justify your construction, that is what is the characteristic property of the Mercedes Benz frame that your frame for $\mathbb{R}^3$ also satisfies?

b.) Write a python function that plots the frame vectors.

c.) Is the frame tight? Justify your answer.

d.) Construct a dual frame for your $\mathbb{R}^3$ analogue of the Mercedes Benz frame.

e.) Write a python function that project a vector into the frame and reconstructs it.

3.) **Biorthogonal bases for function spaces:** A class of polynomials on $[0, 1]$ with many convenient properties are Bernstein polynomials:

$$b_{\nu,n}(x) = \binom{n}{v} x^{\nu} (1-x)^{n-\nu} \tag{3}$$

with the order satisfying $\nu = 0, \cdots, n$.

a.) Implement a function that evaluates Bernstein polynomials. How robust is your function numerically?

b.) Generate plots of all Bernstein polynomials for $n = 0, \cdots, 5$ (the polynomials of the same degree should be in the same plot).

c.) For $n = 3$ show that Bernstein polynomials are not orthogonal but nonetheless provide a basis for the space $\Pi^3([0, 1])$ of polynomials up to degree 3. Can you show this for arbitrary n?

d.) For $n = 3$ construct the dual basis for Bernstein polynomials for $\Pi^3([0, 1])$. For this use Legendre polynomials, rescaled to be defined over $[0, 1]$, as an orthonormal reference basis. Plot the dual basis functions.

4.) **Orthonormal bases for the space of polynomials** $\Pi^n([-1, 1])$**:** An orthonormal basis for the space of polynomials $\Pi^n([-1, 1])$ is given by Legendre polynomials $\{P_i(x)\}_{i=0}^{n-1}$. Construct another orthonormal basis for the space.