Mathematical Tools for Computer Graphics 2017

Tutorial 5

In this tutorial we will consider an important application of the Riesz representation theorem. Specifically, we will study the point evaluation functional $\delta_x(\cdot)$ for function spaces where it is continuous. Such spaces are known as Reproducing Kernel Hilbert Spaces (RKHS) and they play a critical role for interpolation and integration.

1.) Apply the Riesz representation theorem to the point evaluation functional $\delta_{\bar{x}}(\cdot)$ assuming it is continuous in a Hilbert space $\mathcal{H}$.

2.) Consider the Hilbert space $\mathcal{H}_\chi^n([0,1]) \subset L_2([0,1])$ given by

$$\mathcal{H}_\chi^n([0,1]) = \operatorname{span}\Big(\big\{ \chi_i(x) \big\}_{i=1}^n \Big)$$

where $\chi_i(x)$ is the characteristic function

$$\chi_i(x) = \begin{cases} 1 & x \in [(i-1)/n, i/n) \\ 0 & \text{otherwise} \end{cases}$$

equipped with the standard L_2 inner product. Is the point evaluation functional for $\mathcal{H}_\chi^n([0,1])$ continuous?

3.) Let $\mathcal{H}(X)$ be a reproducing kernel Hilbert space, i.e. a Hilbert space where the point evaluation functional is continuous, and $\{\varphi_i\}_{i=1}^n$ an orthonormal basis for the space. Derive the basis expansion for the reproducing kernel $k_{\bar{x}}(x)$ with respect to $\{\varphi_i\}_{i=1}^n$.

4.) Using Numpy, construct a reproducing kernel $k_{\bar{x}}(x)$ for the space $P_5([-1,1])$ spanned by the first five Legendre polynomials (note that Legendre polynomials are a priori *not* orthonormal). Plot the reproducing kernel and verify the reproducing property numerically.

5.) The color of visible light is described by its spectrum, which is a function

$$f : [400 \text{ nm}, 700 \text{ nm}] \to \mathbb{R}$$

that describes the energy at each wavelength in $[400 \text{ nm}, 700 \text{ nm}]$. In computer graphics, the spectrum is typically represented by three discrete values f_r, f_g, f_b, which are associated with red, green, and blue, respectively. Can these three values reproduce f? If yes, how can this be accomplished?